Master's Thesis Summary
Centro de Investigación en Matemáticas (CIMAT)

A cohomological obstruction to the realization of Mapping Class Groups of Surfaces with Marked Points

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The mapping class group of a closed orientable surface S_g of genus g with a finite set of marked points $\mathbf{z} = \{z_1, \dots, z_k\}$ is defined as the quotient group

$$\mathrm{Mcg}(S_g, \mathbf{z}) := \frac{\mathrm{Diff}^+(S_g, \mathbf{z})}{\mathrm{Diff}_0(S_g, \mathbf{z})},$$

where $\mathrm{Diff}^+(S_g, \mathbf{z})$ denotes the group of orientation-preserving diffeomorphisms that preserve the set $\mathbf{z}$, and $\mathrm{Diff}_0(S_g, \mathbf{z})$ is the path component of the identity.

In this context, the lifting problem of mapping class groups asks whether a subgroup $\Gamma \leq \mathrm{Mcg}(S_g, \mathbf{z})$ admits a lift to the diffeomorphism group that makes the following diagram commute.

$$\begin{array}{ccc} & & \mathrm{Diff}^+(S_g, \mathbf{z}) \\ & \overset{\sigma}{\nearrow} & \big\downarrow {\scriptstyle [\cdot]} \\ \Gamma & \hookrightarrow & \mathrm{Mcg}(S_g, \mathbf{z}). \end{array}$$

Following the approach of Bestvina, Church and Souto [BCS13], in this thesis we provide a detailed proof of the following theorem.

Theorem 1. Let S_g be a closed orientable surface of genus $g \geq 2$, and let $\mathbf{z} = \{z_1, \dots, z_k\}$ be a set with $k \geq 1$ marked points. Then no finite-index subgroup of $\mathrm{Mcg}(S_g, \mathbf{z})$ admits a lift to the diffeomorphism group.

The previous theorem is an algebraic consequence of the one-marked-point case. In this setting, there is an embedding $\pi_1(S_g, z) \hookrightarrow \mathrm{Mcg}(S_g, z)$, known as the *point-pushing* map. We therefore consider the following theorem.

Theorem 2. For a closed oriented surface S_g of genus $g \geq 2$ and a marked point $z \in S_g$, the point-pushing map does not admit a lift that makes the diagram

$$\begin{array}{ccc} & & \mathrm{Diff}^+(S_g, z) \\ & \overset{\sigma}{\nearrow} & \big\downarrow {\scriptstyle [\cdot]} \\ \pi_1(S_g, z) & \xhookrightarrow{\text{push}} & \mathrm{Mcg}(S_g, z) \end{array}$$

commute. In particular, $\mathrm{Mcg}(S_g, z)$ does not admit such a lift.

To find an obstruction to such a lift, we study the Euler class of oriented circle bundles over S_g. For an oriented circle bundle $\xi = (\mathbb{S}^1 \hookrightarrow E \to S_g)$ the Euler class $e(\xi) \in H^2(S_g, \mathbb{Z}) \cong \mathbb{Z}$ is a characteristic class whose evaluation on the fundamental class $[S_g]$ gives the Euler number

$$e(\xi)[S_g] \in \mathbb{Z}.$$

The classification of oriented circle bundles [Mor01, Theorem 6.22] and Milnor's formula [Mil58] allow us to study flat circle bundles, i.e., those constructed by the usual action of the fundamental group on the universal cover, together with a prescribed representation $\rho : \pi_1(S_g, z) \to \mathrm{Homeo}^+(\mathbb{S}^1)$. This provides a diagonal action $\pi_1(S_g, z) \curvearrowright \widetilde{S}_g \times \mathbb{S}^1$ and an associated circle bundle

$$E_\rho := \left(\widetilde{S}_g \times \mathbb{S}^1\right) \Big/ \pi_1(S_g, z) \to S_g,$$

that we call a flat circle bundle. The Milnor-Wood inequality can be stated as follows.

Theorem 3 (Milnor-Wood inequality, [Woo71])**.** Let S_g be a closed orientable surface with genus $g \geq 0$, and let $\rho : \pi_1(S_g, z) \to \mathrm{Homeo}^+(\mathbb{S}^1)$ be a representation. The Euler number of the associated flat circle bundle $E_\rho \to S_g$ satisfies the following inequality

$$|e(E_\rho)[S_g]| \leq \begin{cases} 2g-2 = -\chi(S_g) & \text{if } g \geq 1, \\ 0 & \text{if } g = 0. \end{cases}$$

Moreover, if $\rho : \pi_1(S_g, z) \to \mathrm{Homeo}^+(\mathbb{S}^1)$ commutes with the antipodal map $a : \mathbb{S}^1 \to \mathbb{S}^1$, then the following sharper bound holds

$$|e(E_\rho)[S_g]| \leq \begin{cases} g-1 = -\frac{1}{2}\chi(S_g) & \text{if } g \geq 1 \\ 0 & \text{if } g = 0. \end{cases}$$

In this thesis, written in Spanish, we provide detailed proofs of Milnor's formula and the Milnor-Wood inequality. We then combine these results to prove Theorem 2 by contradiction. Assuming that such a lift exists, we construct an annulus bundle $\mathbb{A} \hookrightarrow E \to S_g$ whose boundary consists of two flat circle bundles. The first is the unit tangent bundle, whose Euler number is

$$e(UTS_g)[S_g] = 2 - 2g,$$

while the latter is a linear circle bundle, associated to a representation $\pi_1(S_g, z) \to \mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{Homeo}^+(\mathbb{S}^1)$ by normalization, whose Euler number is bounded in absolute value by $g-1$ by Theorem 3. The differentiability assumption is essential for constructing an action by linear transformations that gives rise to the latter circle bundle.

The fact that both circle bundles arise as boundary components of the same annulus bundle implies that their Euler numbers must be equal, yielding a contradiction.

My contribution

The contribution of my work is the alternative argument explaining why the Euler numbers of the boundary circle bundles should be equal. In my thesis, I approach this question through the classification of fiber bundles. We consider an homotopical equivalence of groups between

$$\mathrm{Homeo}^+(\mathbb{A}) \to \mathrm{Homeo}^+(\mathbb{S}^1),$$

where $\mathrm{Homeo}^+(\mathbb{A})$ denotes the homeomorphism group of orientation preserving homeomorphisms of $\mathbb{A}$ leaving each boundary invariant, but need not fix it pointwise. This induces an homotopical equivalence between their classifying spaces

$$B\mathrm{Homeo}^+(\mathbb{A}) \simeq B\mathrm{Homeo}^+(\mathbb{S}^1).$$

Appealing to the classification of fiber bundles, this equivalence allows us to compare annulus oriented bundles and oriented circle bundles over the same bases and, consequently, to compare the Euler numbers of the two boundary circle bundles.

Additional contents

1. Poster presented at the Colombian Meeting of Geometry and Topology 2026, ECOGyT (poster pdf).
2. Slides for the 59th Congress of the Mexican Mathematical Society (slides pdf).

References

[BCS13] Mladen Bestvina, Thomas Church, and Juan Souto. "Some groups of mapping classes not realized by diffeomorphisms". In: *Comment. Math. Helv.* 88.1 (2013), pp. 205–220. ISSN: 0010-2571,1420-8946. DOI: 10.4171/CMH/283. URL: https://doi.org/10.4171/CMH/283.

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[Mor01] Shigeyuki Morita. *Geometry of Differential Forms*. Vol. 201. Translations of Mathematical Monographs. Translated from the two-volume Japanese original (1997, 1998) by Teruko Nagase and Katsumi Nomizu, Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xxiv+321. ISBN: 0-8218-1045-6. DOI: 10.1090/mmono/201. URL: https://doi.org/10.1090/mmono/201.

[Woo71] John W. Wood. "Bundles with totally disconnected structure group". In: *Comment. Math. Helv.* 46 (1971), pp. 257–273. ISSN: 0010-2571,1420-8946. DOI: 10.1007/BF02566843. URL: https://doi.org/10.1007/BF02566843.